On the Extrapolation of Band-Limited Functions with Energy Constraints

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Abstract—A modification of the algorithm proposed by Papoulis and Gerchberg for extrapolating band-limited functions[1] is suggested to extend its applicability over data corrupted by noise. We assume that energy constraints are known either for the band-limited signal or for the noise.

In addition, the discrete formulation of the iterative algorithm is derived, and the transition from the continuous algorithm to its digital implementation is presented.

I. Introduction

The extrapolation of a band-limited function $f(t)$ for any t in terms of a finite segment $g(t)$ of $f(t)$ is an essential problem in signal analysis. Various algorithms have been proposed for this problem. All of them can, essentially, be classified in three categories: algorithms based on analytic continuation, algorithms using a series expansion in terms of prolate spheroidal functions [1], [7] (see Appendix A), and iterative algorithms based on successive reduction of the mean-squared error [2]-[5]. However, it is well known that this problem is ill-posed, i.e., its solution $f(t)$ is not continuously dependent on $g(t)$, and as Youla [6] says, there exist finite energy perturbations that yield unbounded error. As a result, all the above methods work well if $g(t)$ is exactly a segment of $f(t)$ and there are no errors associated with their numerical implementation. Unfortunately, these requirements are almost never satisfied. For the solutions based on the prolate spheroidal functions expansion, there are available various techniques for stabilizing them. All of them make use of additional knowledge about $f(t)$ beyond its band-limited character. These constraints are usually referred to as "regularizers" of ill-posed problems. In order to obtain such a regularized solution, Slepian [1] assumed that the energy of the unknown signal $f(t)$ is given. Viano [7] derived another approximating solution of $f(t)$ by simultaneously imposing bounds on the energies of the signal $f(t)$ and the data-error. Miller [8] has also considered various energy bound regularizers in an abstract Hilbert space formulation. However, all the above solutions involved the computation and storage of the prolate spheroidal functions, an extremely difficult numerical task. In addition, they are also limited by series truncation errors. Similar results have been presented by Bertero *et al.* [9] and Rushford *et al.* [10]. On the iterative algorithm, Papoulis [4] suggested the early termination of the iteration but the determination of a terminating criterion is not available yet. Youla [6] investigates this iterative algorithm and provides us with the necessary and sufficient conditions for its convergence.

In this work we modify the iterative algorithm [2], [3] to include energy constraints, connecting in this way the regularized solutions obtained in terms of the prolate spheroidal functions, with the iterative algorithm. The suggested iteration is stable even in the presence of noise and converges into a function which is optimum in minimizing some functionals stated in Section II. Three extrapolating problems with different energy constraints are stated in the next section, and their solutions in terms of the prolate spheroidal functions are presented. In Section III we provide a simple iterative algorithm to obtain the solutions in Section II by using only Fourier transformations. This iteration was obtained by a proper modification of the algorithm proposed by Papoulis and Gerchberg, and it invokes a regularization parameter μ. Section IV discusses the numerical evaluation of μ.

Section V is devoted to the solution of the discrete versions of the three extrapolation problems and an equivalent discrete iteration is also derived. The connection between the continuous extrapolation algorithm and its numerical implementation is also discussed in Section V. Finally, the applicability of the suggested iteration is illustrated with a numerical example presented in Section VI.

II. Extrapolation with Energy Constraints

To avoid lengthy formulas we adopt the usual norm notation, denoting hereafter

$$\|f(t)\|^2 = \int_{-\infty}^{\infty} |f(t)|^2 \, dt$$

$$\|f(t)\|_T^2 = \int_{-T}^{T} |f(t)|^2 \, dt.$$

For simplicity, throughout this work time functions shall be assumed to be real, but the results are valid also for complex functions.

Let $f(t)$ be a σ-band-limited function, i.e.,

$$F(\omega) = 0 \quad \text{for} \quad |\omega| > \sigma$$
$$\|f(t)\|^2 < \infty \tag{1}$$

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[1] The band-limited extrapolation algorithm was first presented by A. Papoulis in 1973 in an internal JSTAC report [2]; it was published independently by R. W. Gerchberg in 1974 [3]; the proof of the convergence was given by A. Papoulis in 1975 [4].

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where $F(\omega)$ is the Fourier transform of $f(t)$ and let

$$g(t) = f(t) + n(t) \qquad |t| < T.$$

We want to estimate $f(t)$ in terms of its noisy segment $g(t)$. As we mentioned in the Introduction, there exist errors $n(t)$ with finite but arbitrary small energy that yield arbitrary large errors on the solution $f(t)$ at any point outside $(-T,T)$. However, if additional information is given on $f(t)$ beyond its band-limited property, the estimation of $f(t)$ can be improved and the ill-posed character of the problem be removed. In this work, we will assume that energy bounds on $f(t)$ and/or $n(t)$ are known. Based upon the given additional information, we state three different versions of the extrapolation problem; and the optimum estimator $f_*(t)$ of $f(t)$ is derived in terms of a prolate spheroidal series expansion.

Problem 1: Given that the *energy* of $f(t)$ is equal to or less than R^2, i.e.,

$$\|f(t)\|^2 \leqslant R^2 \tag{2}$$

approximate $f(t)$ with $f_*(t)$ in F_1 such that

$$I_1 = \|f_*(t) - g(t)\|_T^2 = \min_{F_1} \|f(t) - g(t)\|_T^2 \tag{3}$$

where F_1 is the family of σ-band-limited functions $f(t)$ satisfying (2).

This is the same problem considered and solved by Slepian *et al.* [1]. His solution is the same with the one given in Theorem 1.

Problem 2: Given that the energy of the noise in $(-T, T)$ is equal to or less than ϵ^2, i.e., $\|n(t)\|_T^2 \leqslant \epsilon^2$, find an $f_*(t)$ in F_2 such that

$$\|f_*(t) - g(t)\|_T^2 \leqslant \epsilon^2 \tag{4}$$

$$I_2 = \|f_*(t)\|^2 = \min_{F_2} \|f(t)\|^2 \tag{5}$$

where F_2 is the family of σ-band-limited functions $f(t)$ satisfying $\|f(t) - g(t)\|_T^2 \leqslant \epsilon^2$.

Problem 3: Given that the energy of $f(t)$ is equal to or less than R^2 and that the energy of $n(t)$ in $(-T, T)$ is equal to or less than ϵ^2, find a σ-band-limited function $f_*(t)$ such that

$$\|f_*(t) - g(t)\|_T^2 \leqslant \epsilon^2 \tag{6}$$

$$\|f_*(t)\|^2 \leqslant R^2. \tag{7}$$

Problem 3 has been considered also by Viano [7], but he obtained an approximating solution. Finally, Miller [8] has also considered all the above three cases in an abstract Hilbert space and derived similar solutions using the method of eigenfunction expansion [1], but the form of his solutions is different from ours.

Theorem 1: Let

$$g(t) = \sum_{k=0}^{\infty} b_k \phi_k(t) \qquad \text{for} \qquad |t| < T \tag{8}$$

where $\phi_k(t)$ are the prolate spheroidal functions (see A1) and b_k defined as in (A8). Then a solution to Problems 1-3 is of the form

$$f_*(t) = \sum_{k=0}^{\infty} \frac{\lambda_k b_k}{\lambda_k + \mu} \phi_k(t). \tag{9}$$

a) For Problem 1, $\mu = 0$ or $\mu = \mu_R > 0$ where μ_R is the solution of

$$R(\mu_R) = \sum_{k=0}^{\infty} \frac{\lambda_k^2 b_k^2}{(\lambda_k + \mu_R)^2} = R^2. \tag{10}$$

b) For Problem 2, $\mu = \infty$ or $0 \leqslant \mu = \mu_\epsilon < \infty$ where μ_ϵ is the solution of

$$J(\mu_\epsilon) = \sum_{k=0}^{\infty} \frac{\lambda_k \mu_\epsilon^2 b_k^2}{(\lambda_k + \mu_\epsilon)^2} = \epsilon^2. \tag{11}$$

c) For Problem 3,

$$0 \leqslant \mu_R \leqslant \mu \leqslant \mu_\epsilon. \tag{12}$$

If $\mu_\epsilon < \mu_R$, then Problem 3 does not have a solution. μ_ϵ and μ_R are the solutions of $R(\mu_R) = R^2$ and $J(\mu_\epsilon) = \epsilon^2$, respectively.

Proof:

a) With $f(t)$ a σ-band-limited function (1), we obtain from (A6)

$$f(t) = \sum_{k=0}^{\infty} a_k \phi_k(t). \tag{13}$$

Using relations (8), (13), and (A9), we get

$$I = \|f(t) - g(t)\|_T^2 = \sum_{k=0}^{\infty} \lambda_k (b_k - a_k)^2. \tag{14}$$

Thus, Problem 1 is equivalent to determine a_k, minimizing I under the constraint

$$\sum_{k=0}^{\infty} a_k^2 \leqslant R^2. \tag{15}$$

Using Lagrangian multipliers we have the unconstrained minimization of

$$H(a_k, \mu) = \sum_{k=0}^{\infty} \lambda_k (b_k - a_k)^2 + \mu \sum_{k=0}^{\infty} a_k^2. \tag{16}$$

Setting

$$\frac{\partial H(a_k, \mu)}{\partial a_k} = 0, \tag{17}$$

we obtain

$$a_k = \frac{\lambda_k b_k}{\lambda_k + \mu}. \tag{18}$$

Substituting (18) into (14) and (15), we have to determine μ such that

$$R(\mu) = \sum_{k=0}^{\infty} \frac{\lambda_k^2 b_k^2}{(\lambda_k + \mu)^2} \leqslant R^2 \tag{19}$$

and minimizing

$$J(\mu) = \sum_{k=0}^{\infty} \frac{\lambda_k \mu^2 b_k^2}{(\lambda_k + \mu)^2}. \tag{20}$$

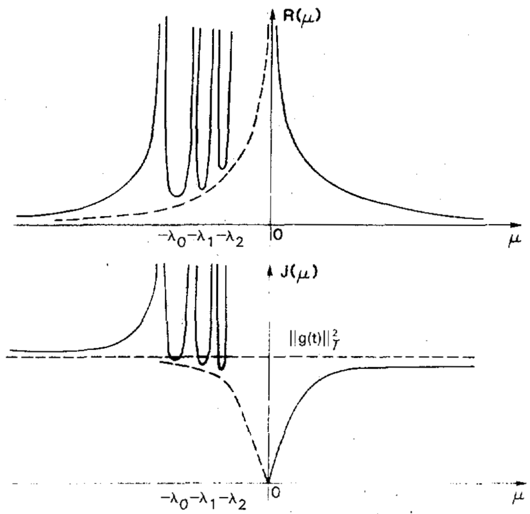


Fig. 1. Sketch of $R(\mu)$ and $J(\mu)$.

It can be shown that since for any positive $\mu, J(-\mu) > J(\mu)$ and $R(-\mu) > R(\mu)$, we need to consider only $\mu \geqslant 0$. Since for $\mu \geqslant 0$, $R(\mu)$ and $J(\mu)$ (Section IV) are, respectively, monotonically decreasing and increasing functions of μ (see Fig. 1), we have

$$\mu = \begin{cases} \mu_R > 0 & \text{if } R^2 < \sum_{k=0}^{\infty} b_k^2 \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

where μ_R is the positive root of $R(\mu) = R^2$.

b) Following a procedure which is similar to a), we obtain that the μ for Problem 2 is

$$\mu = \begin{cases} \mu_\epsilon \geqslant 0 & \text{if } \epsilon^2 < \|g(t)\|_T^2 \\ \infty & \text{otherwise} \end{cases} \tag{22}$$

where μ_ϵ is the positive root of $J(\mu) = \epsilon^2$.

c) If $f_*(t)$ is a solution of Problem 3, then

$$J(\mu_R) = \min_{F_1} \|f(t) - g(t)\|_T^2 \leqslant \|f_*(t) - g(t)\|_T^2 \leqslant \epsilon^2 = J(\mu_\epsilon), \tag{23}$$

because $f_*(t) \epsilon F_1$. Therefore, since $J(\mu)$ is monotonically increasing, the necessary condition for the existence of a solution to Problem 3 is

$$\mu_\epsilon \geqslant \mu_R .$$

It is easy to find that any μ in the closed interval $[\mu_R, \mu_\epsilon]$ will provide us with an acceptable $f_*(t)$.

The form obtained in (9) is usually referred to as a "regularized" solution [12] of the original ill-posed extrapolation problem because it is defined for any given segment $g(t)$ and in the absence of noise, i.e., $\mu = 0, f_*(t) = f(t)$. μ is called the regularization parameter.

Also, we would like to comment on the meaning of $\mu = \infty$ for Problem 2. If $\|g(t)\|_T^2 \leqslant \epsilon^2$, then it is clear that the trivial solution $f_*(t) = 0$ is the optimum one because it is σ-band-limited, satisfies condition 4, and possesses minimum energy in F_2.

The numerical evaluation of the derived optimum solution $f_*(t)$ presents two difficulties. The first one is the use of the prolate spheroidal functions and the other one is the calculation of the parameter μ from (21) and (22). These two problems are addressed in Sections III and IV.

III. The Iterative Solution

To avoid the complexity associated with the eigenfunctions expansion, i.e., storing $\phi_k(t)$, evaluation of λ_k, etc., we shall devise an iteration to converge into $f_*(t)$ [see (9)], assuming that μ is given.

Let

$$f_{n+1}(t) = [(1 - \alpha\mu) f_n(t) + \alpha(g(t) - f_n(t)) P_T(t)] * \frac{\sin \sigma t}{\pi t} \tag{24}$$

$$f_0(t) = 0$$

where

$$P_T(t) = \begin{cases} 1 & |t| < T \\ 0 & |t| > T \end{cases}$$

and $*$ denotes convolution.

Then, with

$$0 < \alpha < \frac{2}{1+\mu}$$

$$\lim_{n \to \infty} f_n(t) = f_*(t).$$

Proof: Because $f_n(t)$ is σ-band-limited,

$$f_n(t) = \sum_{k=0}^{\infty} a_{k,n} \phi_k(t). \tag{25}$$

Substituting $f_n(t)$ and $f_{n+1}(t)$ from (25) and $g(t)$ from (8) into (24), and using (A1) and (A2), we readily obtain, by equating the coefficients of $\phi_k(t)$,

$$a_{k,n+1} = [1 - \alpha(\mu + \lambda_k)]\, a_{k,n} + \alpha \lambda_k b_k \tag{26}$$

which is a recursive equation in $a_{k,n}$ with $a_{k,0} = 0$. Its solution is

$$a_{k,n} = \frac{\lambda_k b_k}{\lambda_k + \mu} [1 - (1 - \alpha(\lambda_k + \mu))^n]. \tag{27}$$

Setting

$$0 < \alpha < \frac{2}{1+\mu} < \frac{2}{\lambda_k + \mu} \tag{28}$$

we obtain [see (A7) and (27)]

$$\lim_{n \to \infty} \|f_n(t) - f_*(t)\|^2 = \lim_{n \to \infty} \sum_{k=0}^{\infty} \left(\frac{\lambda_k b_k}{\lambda_k + \mu}\right)^2 (1 - \alpha(\lambda_k + \mu))^{2n} = 0 \tag{29}$$

which results in

$$\lim_{n\to\infty} f_n(t) = f_*(t) \tag{30}$$

because for the σ-band-limited function, $f_n(t) - f_*(t)$ [11, p. 169],

$$\max |f_n(t) - f_*(t)| \leqslant \sqrt{\frac{\sigma}{\pi}}\, \|f_n(t) - f_*(t)\|.$$

This completes the proof because of (29).

Notes

1) The constant α is usually referred to as a relaxation parameter. It is used to guarantee the convergence of the iteration. If monotonic convergence, i.e., $\|f_n(t)\| > \|f_{n-1}(t)\|$, is desirable, then the condition $0 < \alpha \leqslant 1/1+\mu$ is sufficient. This can be easily obtained from (27) because

$$\|f_n(t)\|^2 = \sum_{k=0}^{\infty} |a_{k,n}|^2 > \sum_{k=0}^{\infty} |a_{k,n-1}|^2 = \|f_{n-1}(t)\|^2.$$

2) For $\mu = 0$ and $\alpha = 1$ the iteration in (24) becomes the Papoulis–Gerchberg algorithm. This iteration converges if

$$\sum_{k=0}^{\infty} b_k^2 < \infty,$$

which is equivalent to saying that the noise $n(t)$ in (1) is a segment of a σ-band-limited function.

The significance of the parameter μ in the iteration can also be described in terms of integral equations.

If we rearrange the terms in (24), and make use of the fact that $f_n(t)$ is σ-band-limited, we obtain

$$f_{n+1}(t) = (1 - \alpha\mu) f_n(t) + \alpha \int_{-T}^{T} (g(\tau) - f_n(\tau)) \frac{\sin \sigma(t-\tau)}{\pi(t-\tau)}\, d\tau. \tag{31}$$

Because $f_n(t) \to f_*(t)$ uniformly (30), taking limits for $n \to \infty$ on both sides of (31), we have that $f_*(t)$ should satisfy

$$\mu f_*(t) + \int_{-T}^{T} f_*(\tau) \frac{\sin \sigma(t-\tau)}{\pi(t-\tau)}\, d\tau = \int_{-T}^{T} g(\tau) \frac{\sin \sigma(t-\tau)}{\pi(t-\tau)}\, d\tau. \tag{32}$$

But this is a Fredholm integral equation of the *second* kind, and it is well known [12] that the solution $f_*(t)$ of this equation is well behaved. It can be shown also directly from (9) that $f_*(t)$ is the solution of (32) and then the algorithm in (24) can be simply considered as the successive approximation solution of (32).

Depending on the numerical implementation of the iterative solution, (31) may be a more preferable form than (24), because it avoids the use of infinite integrals. If $\alpha = 1$ and $\mu = 0$, (31) becomes the iteration used by Cadzow in [13]. Notice that in this case the solution $f_*(t)$ satisfies a Fredholm integral equation of the *first* kind.

$$\int_{-\infty}^{\infty} f_*(\tau) \frac{\sin \sigma(t-\tau)}{(t-\tau)}\, d\tau = f(t) \qquad t\epsilon[-T, T]. \tag{33}$$

The following corollary shows the stability of the iterative solution (24).

Corollary 1: For any $\mu > 0$ and $0 < \alpha \leqslant \dfrac{2}{1+\mu}$

$$\|f_n(t)\|^2 < \frac{\|g(t)\|_T^2}{\mu^2}. \tag{34}$$

Proof: From (27) we conclude that

$$\|f_n(t)\|^2 < \|f_*(t)\|^2$$

and (34) results from the fact that

$$\|f_*(t)\|^2 = \sum_{k=0}^{\infty} \frac{\lambda_k^2 b_k^2}{(\lambda_k + \mu)^2} < \sum_{k=0}^{\infty} \frac{\lambda_k^2 b_k^2}{\mu^2} < \frac{1}{\mu^2} \sum_{k=0}^{\infty} \lambda_k b_k^2 = \frac{1}{\mu^2} \|g(t)\|_T^2$$

where for the inequalities we used that $0 < \lambda_k < 1$ and for the equality (A9). Actually, we can also prove that

$$\|f_n(t)\|^2 < \frac{\|g(t)\|_T^2}{2\mu}. \tag{35}$$

IV. On the Evaluation of μ_R and μ_ϵ

The evaluation of μ_R and μ_ϵ is the second problem we have to consider. Relations (10) or (11) cannot be used because they require knowledge of λ_k and b_k. To avoid this problem we use the iteration to obtain $R(\mu)$ or $J(\mu)$ for various values of μ and then utilize one of the various searching numerical techniques (for example, bisection or secant, etc.) to find the solution of (11) or (12). Below we present some theorems which can be used to speed up the evaluation of μ_R or μ_ϵ.

Lemma 1: Let $f_*(t, \mu)$ denote the limit of the iteration (24) with a given μ, then

$$\|f_*(t,\mu)\|_T^2 + 2\mu R(\mu) + J(\mu) = \|g(t)\|_T^2. \tag{36}$$

Proof: Using relations (19), (20), and

$$\|f_*(t,\mu)\|_T^2 = \sum_{k=0}^{\infty} \frac{\lambda_k^3 b_k^2}{(\lambda_k + \mu)^2},$$

we obtain

$$\|f_*(t,\mu)\|_T^2 + 2\mu R(\mu) + J(\mu) = \sum_{k=0}^{\infty} (\lambda_k^3 + 2\mu\lambda_k^2 + \lambda_k \mu^2) \frac{b_k^2}{(\lambda_k+\mu)^2} = \sum_{k=0}^{\infty} \lambda_k b_k^2 = \|g(t)\|_T^2.$$

Q.E.D.

Lemma 2: The functions $R(\mu)$ and $J(\mu)$ for $\mu \geqslant 0$ are monotonically decreasing and increasing, respectively.

Proof: Taking the derivatives we can easily show (19), (20) that for $\mu > 0$,

$$R'(\mu) = -2 \sum_{k=0}^{\infty} \frac{\lambda_k^2 b_k^2}{(\lambda_k + \mu)^3} < 0 \qquad \mu > 0$$

$$J'(\mu) = 2 \sum_{k=0}^{\infty} \frac{\mu \lambda_k^2 b_k^2}{(\lambda_k + \mu)^3} > 0 \qquad \mu > 0.$$

Theorem 3: The constants μ_R and μ_ϵ satisfy the following bounds:

$$0 \leqslant \mu_R < \frac{\|g(t)\|_T^2}{2R^2} \tag{37}$$

$$0 \leqslant \mu_\epsilon < \frac{\epsilon}{\|g(t)\|_T - \epsilon} \quad \text{if} \quad \|g(t)\|_T^2 > \epsilon^2. \tag{38}$$

Proof: The positivity of μ_R and μ_ϵ has been proved in Theorem 1. The upper bound of (37) is obtained directly from (36) because $R(\mu_R) = R^2$. From (36) we also obtain, assuming $\mu < \infty$,

$$\frac{\|g(t)\|_T^2 - J(\mu)}{1 + 2\mu} < R(\mu) < \sum_{k=0}^{\infty} \frac{\lambda_k b_k^2}{(\lambda_k + \mu)^2} = \frac{J(\mu)}{\mu^2}$$

or

$$\mu < \frac{\sqrt{J(\mu)}}{\|g(t)\|_T - \sqrt{J(\mu)}}$$

and (38) follows because $J(\mu_\epsilon) = \epsilon^2$.

Theorem 3 provides us with an initial value μ_0 for μ_ϵ or μ_R. Using the iteration we can evaluate $R(\mu_0) = \|f_*(t, \mu_0)\|^2$ or $J(\mu_0) = \|g(t) - f_*(t, \mu_0)\|_T^2$; and utilizing the monotonicity of $R(\mu)$ and $J(\mu)$, we can evaluate the next value μ_1 by using various numerical methods for solving nonlinear equations.

Theorem 4: With $0 < \alpha \leqslant \dfrac{1}{1 + \mu}$ (39)

$$R(\mu)\left[1 - \left(\frac{1}{1 + \mu}\right)^n\right]^2 < \|f_n(t)\|^2 < R(\mu) \tag{40}$$

$$J(\mu) \leqslant \|f_n(t) - g(t)\|_T^2 < J(\mu)\left[1 + \frac{1}{\mu}(1 - \alpha\mu)^n\right]^2. \tag{41}$$

Proof: From (27) we obtain

$$\|f_n(t)\|^2 = \sum_{k=0}^{\infty} \frac{\lambda_k^2 b_k^2}{(\lambda_k + \mu)^2} [1 - \alpha(\lambda_k + \mu))^n]^2. \tag{42}$$

Using (39) and $0 < \lambda_k < 1$ we readily obtain

$$1 > (1 - (1 - \alpha(\lambda_k + \mu))^n)^2 > \left(1 - \left(1 - \frac{\lambda_k + \mu}{1 + \mu}\right)^n\right)^2$$

$$= \left[1 - \left(\frac{1 - \lambda_k}{1 + \mu}\right)^n\right]^2 > \left[1 - \left(\frac{1}{1 + \mu}\right)^n\right]^2,$$

and (40) follows from the above and (19). Similarly, we can prove (41) because

$$\|f_n(t) - g(t)\|_T^2 = \sum_{k=0}^{\infty} \frac{\lambda_k b_k^2 \mu^2}{(\lambda_k + \mu)^2}\left(1 + \frac{\lambda_k}{\mu}(1 - \alpha(\lambda_k + \mu))^n\right)^2. \tag{43}$$

Relation (40) can reduce significantly the number of calculations in Problem 1, because

$$\text{if } \|f_n(t)\|^2 > R^2, \quad \text{then} \quad \mu > \mu_R$$

and

$$\text{if } \|f_n(t)\|^2\, R^2\left(1 - \left(\frac{1}{1 + \mu}\right)^n\right)^2, \quad \text{then} \quad \mu < \mu_R. \tag{44}$$

Similarly, using (41) we have for Problem 2

$$\text{if } \|f_n(t) - g(t)\|_T^2 < \epsilon^2, \quad \text{then} \quad \mu < \mu_\epsilon$$

and

$$\text{if } \|f_n(t) - g(t)\|_T^2 > \epsilon^2\left(1 + \frac{1}{\mu}(1 - \alpha(\lambda_k + \mu))^n\right)^2, \quad \text{then} \quad \mu > \mu_\epsilon. \tag{45}$$

Finally, (40) and (41) can be used simultaneously to help us determine an acceptable value of μ for Problem 3.

The application of the above statements will be demonstrated in Section VI with a numerical example.

V. The Discrete Problem

There recently has developed a controversy over whether analog algorithms can be digitized, and over the connection between the results of a numerical implementation versus the analog solution. For discrete signals the analyticity property vanishes due to sampling, and extrapolated estimate need not coincide with the original one. However, if the energy bounds, described in Section II, are imposed, the uniqueness of the solution is restored and the connection between the analog solution and its discrete digital implementation can be established.

In this section we will restate the problems of Section II in a discrete environment and rederive their solutions. In Section V-A we will consider the problem of extrapolating a band-limited sequence $f[n]$ for any n, in terms of a finite set of noisy samples. This problem is obtained when a σ-band-limited function $f(t)$ is sampled with a sampling interval $T_s \leqslant \pi/\sigma$, and we want to estimate its Fourier transform in terms of a finite set of samples $f[n] = f(nT_s)$ $|n| \leqslant M$. In Section V-B the problem of extrapolating *N-periodic* band-limited sequences will be presented and its solution under the equivalent energy constraints will be given. This situation arises every time the environment of a computer is used. Digitization and roundoff errors can be considered as part of our noise term. Finally, in Section V-C the relation between the analog algorithm and its numerical realization is derived.

Also, we would like to mention that in addition to the importance of this section to the numerical implementation of the analog algorithm, there are cases where the problems are phrased directly in their *discrete form* (cases A and B), i.e., we are asked to extrapolate periodic functions, or trigonometric polynomials in terms of their samples.

Prolate spheroidal functions (PSF) were the basic tool of the analog solution in Section II. Similarly, the key function for the two discrete problems will be the discrete prolate spheroidal sequences (DPSS) and the periodic discrete prolate spheroidal sequence (P-DPSS). Their basic properties and definitions are stated in Appendixes B and C. For a more detailed description of DPSS see Slepian [14] or Papoulis [15], and for P-DPSS see [16] and [17].

Throughout this work the usual norm notation for sequences is used, denoting hereafter

$$\|f[n]\|^2 = \sum_{n=-\infty}^{\infty} |f[n]|^2$$

$$\|f[n]\|^2_{M,N} = \sum_{n=M}^{N} |f[n]|^2.$$

A. Extrapolating Band-Limited Sequences with Energy Constraints

Let $f[n]$ be a σ-band-limited sequence, i.e.,

$$F(\theta) = \sum_{n=-\infty}^{\infty} f[n]e^{-jn\theta} = 0 \qquad \pi > |\theta| > \sigma$$

$$\|f[n]\|^2 < \infty$$

where $F(\theta)$ is the discrete Fourier transform of $f[n]$ and let

$$g[n] = f[n] + \eta[n] \qquad |n| \leqslant M$$

be the given data. We want to estimate $f[n]$, for all n, in terms of its noisy segment $g[n]$.

It should be pointed out again that in constrast with the continuous case, $f[n]$ cannot be determined uniquely from $g[n]$ in the absence of noise. However, imposing the discrete version of the constraints (2), (3), or (4) and (5), the uniqueness of the solution can be reestablished, and a theorem equivalent to Theorem 1 will be derived.

Lemma 3: Let $h[m]$ be known for $|m| \leqslant M$. Then the function $f^+[m]$ satisfying the conditions

$f^+[m]$ is σ-band-limited

$$f^+[m] = h[m] \qquad |m| \leqslant M$$

and $\|f^+[m]\|^2 = \min \{\|f[m]\|^2;\ f[m]$ is σ-band-limited and $f[m] = h[m]\ |m| \leqslant M\}$ is given by

$$f^+[m] = \sum_{k=0}^{2M} a_k \phi_k[m] \tag{46}$$

where $\phi_k[m]$ are the DPSS described in Appendix B, and

$$a_k = \frac{1}{\lambda_k} \sum_{m=-M}^{M} h[m]\phi_k[m].$$

Proof: Let $f[m]$ be a σ-band-limited sequence and $f[m] = h[m]$ for $|m| \leqslant M$. Then, from (B6)

$$\begin{aligned} f[m] &= \sum_{k=0}^{\infty} a_k\phi_k[m] \\ &= \sum_{k=0}^{2M} a_k\phi_k[m] + \sum_{k=2M+1}^{\infty} a_k\phi_k[m] \\ &= f^+[m] + f_1[m] \end{aligned} \tag{47}$$

where $\phi_k[m]$, $k = 0, 1, \cdots, 2M$ are the DPSS, but

$$f_1[m] = 0 \qquad \text{for} \qquad |m| \leqslant M$$

[see (B8)], thus,

$$f[m] = f^+[m] \qquad |m| \leqslant M \tag{48}$$

and

$$\|f^+[m]\|^2 \leqslant \|f[m]\|^2. \qquad \text{Q.E.D.}$$

This result is used also in [17] where $f^+[m]$ is referred to as the "minimum norm least square solution."

Definition: The linear $(2M + 1)$-dimensional subspace determined by $\phi_k[m]$, $k = 0, 1, \cdots, 2M$, where $\phi_k[m]$ are the $(2M + 1)$ DPSS, is denoted by F_3.

Using Lemma 3, the space of σ-band-limited sequences in Problems 1-3 can be replaced by F_3. Then the uniqueness of the solution has been restored and all the proofs in Sections II–IV can easily be duplicated for the σ-band-limited sequences case; and the equivalent results are obtained by *merely substituting* ∞ *by* $2M$ and $f(t)$ by $f[m]$ in Sections II-IV. For example, $f_*[m]$, the equivalent of $f_*(t)$ in Theorem 1, is

$$f_*[m] = \sum_{k=0}^{2M} \frac{\lambda_k b_k}{\lambda_k + \mu} \phi_k[m] \tag{49}$$

where $\phi_k[m]$ are the DPSS [see (B1), and b_k is defined as in (B8)]. At this point, it should be pointed out that the special case of Problem 2 with $\epsilon = 0$ has been also considered and solved in [17].

The equivalent discrete form of the iterative solution is easily found to be

$$f_{n+1}[m] = [(1 - \alpha\mu)f_n[m] + \alpha(g[m] - f_n[m])P_M[m]] * \frac{\sin \sigma m}{\pi m} \tag{50}$$

with $f_0[m] = 0$ where $*$ denotes discrete convolution, and

$$P_M[m] = \begin{cases} 0 & |m| > M \\ 1 & |m| \leqslant M. \end{cases} \tag{51}$$

An easier way to implement (49) [see (31)] is

$$f_{n+1}[m] = (1 - \alpha\mu)f_n[m] + \alpha \sum_{l=-M}^{M} (g[l] - f_n[l]) \frac{\sin \sigma(m - l)}{\pi(m - l)} \tag{52}$$

because it avoids the infinite summation. Similar to (32), the solution $f_*[m]$ in (49) satisfies the equation

$$\mu f_*[m] + \sum_{l=-M}^{M} f_*[l] \frac{\sin \sigma(m-l)}{\pi(m-l)} = \sum_{l=-M}^{M} g[l] \frac{\sin \sigma(m-l)}{\pi(m-l)}. \tag{53}$$

Besides the iterative algorithms mentioned above [(50) or (52)], (53) can be solved in the following way, which is similar to that provided in [17] and [18] for the noiseless case.

Denote

$$\delta[m-l] = \begin{cases} 1 & m = l \\ 0 & m \neq l. \end{cases}$$

First, solve the $(2M+1)$ Toeplitz equations

$$\sum_{l=-M}^{M} \left[\mu\delta[m-l] + \frac{\sin \sigma(m-l)}{\pi(m-l)}\right] [g[l] - f_*[l]] = \mu g[m] \qquad |m| \leqslant M \tag{54}$$

and determine $f_*[l]$ for $|l| \leqslant M$. Then obtain the solution $f_*[m]$ for any m by using

$$f_*[m] = \frac{1}{\mu} \sum_{l=-M}^{M} [g[l] - f_*[l]] \frac{\sin \sigma(m-l)}{\pi(m-l)}. \tag{55}$$

Compare to the noiseless case [17], [18], where the corresponding Toeplitz matrix is

$$\left[\frac{\sin \sigma(m-l)}{\pi(m-l)}\right]_{m,l=-M,\cdots,0,\cdots,M}$$

which is an ill-conditioned matrix when M is large. In (54), however, the ill-conditioned character of the matrix is improved because of the addition of the constant μ to its diagonal elements. This is a common technique for stabilizing an ill-conditioned matrix; however, here μ is related to the imposed energy bounds on $f[m]$ and/or $\eta[m]$.

B. Extrapolating Periodic Band-Limited Sequence with Energy Constraints

Let $f[n]$ be a K-band-limited N-periodic sequence, i.e., $f[n] = f[n+N]$ and

$$F[m] = \frac{1}{N} \sum_{n=0}^{N-1} f[n] \exp(-j2\pi mn/N) = 0 \qquad K < m < N - K \tag{56}$$

where $F[m]$ is the discrete Fourier series of the N-periodic sequence $f[n]$, and let

$$g[n] = f[n] + \eta[n] \qquad |n| \leqslant M$$

be the given data, where $2M + 1 < N$. We want to estimate the N-periodic sequence $f[n]$, in terms of its noisy segment $g[n]$.

Here we should examine the cases of $K \geqslant M$ and $K < M$ separately.

i) For the case $K \geqslant M$, all the statements will be similar to those in Section V-A. For example, the equivalent solution of Theorem 1 is

$$f_*[m] = \sum_{i=0}^{2M} \frac{\lambda_i b_i}{\lambda_i + \mu} \phi_i[m] \tag{57}$$

where $\phi_i[m]$ are the P-DPSS [see (C1)] and b_i is defined as in (C12).

The iterative solution is

$$f_{n+1}[m] = \sum_{l=l_0}^{l_0+N-1} [(1-\alpha\mu) f_n[l] + \alpha(g[l] - f_n[l]) P_M[l]] \frac{\sin \frac{\pi}{N}(2K+1)(m-l)}{N \sin \frac{\pi}{N}(m-l)} \tag{58}$$

with

$f_0[m] = 0$

or

$$f_{n+1}[m] = (1-\alpha\mu) f_n[m] + \alpha \sum_{l=-M}^{M} (g[l] - f_n[l]) \frac{\sin \frac{\pi}{N}(2K+1)(m-l)}{N \sin \frac{\pi}{N}(m-l)} \tag{59}$$

with

$f_0[m] = 0.$

The solution $f_*[m]$ satisfies the equation

$$\mu f_*[m] + \sum_{l=-M}^{M} f_*[l] \frac{\sin \frac{\pi}{N}(2K+1)(m-l)}{N \sin \frac{\pi}{N}(m-l)} = \sum_{l=-M}^{M} g[l] \frac{\sin \frac{\pi}{N}(2K+1)(m-l)}{N \sin \frac{\pi}{N}(m-l)}. \tag{60}$$

ii) For the case $K < M$, see (C14),

$$g[m] = \sum_{i=0}^{2K} b_i \phi_i[m] + \tilde{g}[m] \qquad |m| \leqslant M \tag{61}$$

where $\phi_i[m]$ are the P-DPSS with nonzero eigenvalues and the term $\tilde{g} = (g[-M], \cdots, g[M])$ is orthogonal to $\phi_i = (\phi_i[-M], \cdots, \phi_i[M])$, $i = 0, \cdots, 2K$. For the extrapolating Problem 1, the solution is

$$f_*[m] = \sum_{i=0}^{2K} \frac{\lambda_i b_i}{\lambda_i + \mu} \phi_i[m]$$

with μ the solution of

$$R(\mu) = \sum_{i=0}^{2K} \frac{\lambda_i^2 b_i^2}{(\lambda_i + \mu)^2} = R^2.$$

For Problem 2, there is no solution if $\|\tilde{g}[m]\|^2_{-M,M} > \epsilon^2$. When $\|\tilde{g}[m]\|^2_{-M,M} \leq \epsilon^2$, the solution is

$$f_*[m] = \sum_{i=0}^{2K} \frac{\lambda_i b_i}{\lambda_i + \mu} \phi_i[m] \tag{62}$$

where $\mu = \infty$ if $\epsilon^2 = \|\tilde{g}[m]\|^2_{-M,M}$ or μ satisfies the equation

$$J(\mu) \triangleq \sum_{i=0}^{2K} \frac{\lambda_i \mu^2 b_i^2}{(\lambda_i + \mu)^2} = \epsilon^2 - \|\tilde{g}[m]\|^2_{-M,M} \quad \text{for } \epsilon^2 > \|\tilde{g}[m]\|^2_{-M,M}. \tag{63}$$

Denote the solution of (63) by $\mu_{\epsilon'}$ where

$$\epsilon' = \sqrt{\epsilon^2 - \|\tilde{g}[m]\|^2_{-M,M}}.$$

Substituting μ_ϵ by $\mu_{\epsilon'}$, the assertion which is similar to c) in Theorem 1 can be obtained.

When $K < M$, (58) and (59) as well as (60) can also be derived assuming that the solution exists.

C. Digital Implementation of the Analog Algorithm

Now we will discuss the connection between the analog solution of the algorithm and the solution of its digital implementation. Cadzow [13] has also studied this problem, but only for the noiseless case.

Let $\mu > 0$ and $g(t)$ given for $|t| \leq T$. If we sample $g(t)$ with a sampling interval $T_s \leq \pi/\sigma$, we obtain the discrete data

$$g[l] = g(lT_s) \qquad l = -M, \cdots, M, \qquad M = T/T_s. \tag{64}$$

Let us choose N large enough such that if

$$K = \text{integer}\left[\frac{(N-1)}{2} \frac{\sigma T_s}{\pi}\right]$$

then $N > 2M + 1$ and $K > M$. Applying the iterations described in (58) or (59) on the given data $g[l]$, an N-periodic sequence $f_*^{N,T_s}(mT_s)$ is obtained satisfying the equation

$$\mu f_*^{N,T_s}(mT_s) + \sum_{l=-M}^{M} f_*^{N,T_s}(lT_s) \frac{\sin \frac{\pi}{N}(2K+1)(m-l)}{N \sin \frac{\pi}{N}(m-l)} = \sum_{l=-M}^{M} g(lT_s) \frac{\sin \frac{\pi}{N}(2K+1)(m-l)}{N \sin \frac{\pi}{N}(m-l)}$$

$$m = -\frac{N}{2}, \cdots, \frac{N}{2} - 1. \tag{65}$$

For simplicity, N is assumed to be even.

For fixed T_s, let N tend to infinity. Since

$$\lim_{N\to\infty} \frac{\sin \frac{\pi}{N}(2K+1)(m-l)}{N \sin \frac{\pi}{N}(m-l)} = \frac{\sin \sigma T_s(m-l)}{\pi(m-l)}, \tag{66}$$

if we define

$$\bar{f}_*^{N,T_s}(mT_s) = \begin{cases} f_*^{N,T_s}(mT_s) & m = \frac{N}{2}, \cdots, \frac{N}{2} - 1 \\ 0 & \text{otherwise,} \end{cases} \tag{67}$$

then, for sufficiently large N, $\bar{f}_*^{N,T_s}(mT_s)$ is an approximation of $f_*^{T_s}(mT_s)$, which satisfies the equation

$$\mu f_*^{T_s}(mT_s) + \sum_{l=-M}^{M} f_*^{T_s}(lT_s) \frac{\sin \sigma T_s(m-l)}{T_s \pi(m-l)} T_s = \sum_{l=-M}^{M} g(lT_s) \frac{\sin \sigma T_s(m-l)}{T_s \pi(m-l)} T_s \qquad |m| = 0, 1, \cdots. \tag{68}$$

Now let T_s tend to zero. Based on the theory of approximate solution for Fredholm integral equation of the second kind [19], $f_*^{T_s}(mT_s)$ will be an approximate solution of (32) when T_s is sufficiently small.

In short, choosing appropriate T_s and N and using iteration (58), the solution of (32) can be derived approximately.

Let us denote respectively with $R^{(i)}(\mu)$ and $J^{(i)}(\mu)$, $i = 1, 2, 3$ the $R(\mu)$ and $J(\mu)$ of the σ-band-limited functions, of the $T_s\sigma$-band-limited sequences, and of the $[(N - 1/2)(\sigma T_s/\pi)]$-band-limited periodic sequences.

Theorem 5:

i) $\lim_{N\to\infty} J^{(3)}(\mu) = J^{(2)}(\mu)$ (69)

ii) $\lim_{N\to\infty} R^{(3)}(\mu) = R^{(2)}(\mu)$ (70)

iii) $\lim_{T_s\to 0} J^{(2)}(\mu) T_s = J^{(1)}(\mu)$ (71)

iv) $\lim_{T_s\to 0} R^{(2)}(\mu) T_s = R^{(1)}(\mu)$. (72)

Proof: From (65), it follows that

$$R^{(3)}(\mu) = \sum_{m=-(N/2)}^{(N/2)-1} |f_*^{N,T_s}(mT_s)|^2$$

$$= \frac{1}{\mu^2} \sum_{m=-(N/2)}^{(N/2)-1} \sum_{l=-M}^{M} \sum_{n=-M}^{M} \cdot \left[g(lT_s) - f_*^{N,T_s}(lT_s)\right]\left[g(nT_s) - f_*^{N,T_s}(nT_s)\right] \cdot \frac{\sin \frac{\pi}{N}(2K+1)(l-m)}{N \sin \frac{\pi}{N}(l-m)} \frac{\sin \frac{\pi}{N}(2K+1)(m-n)}{N \sin \frac{\pi}{N}(m-n)}$$

$$= \frac{1}{\mu^2} \sum_{l=-M}^{M} \sum_{n=-M}^{M} \left[g(lT_s) - f_*^{N,T_s}(lT_s)\right] \cdot \left[g(nT_s) - f_*^{N,T_s}(nT_s)\right] \frac{\sin \frac{\pi}{N}(2K+1)(l-n)}{N \sin \frac{\pi}{N}(l-n)}.$$

Therefore,

$$\lim_{N\to\infty} R^{(3)}(\mu) = \frac{1}{\mu^2} \sum_{l=-M}^{M} \sum_{n=-M}^{M} [g(lT_s) - f_*^{T_s}(lT_s)] \cdot [g(nT_s) - f_*^{T_s}(nT_s)] \frac{\sin \sigma T_s(l-n)}{\pi(l-n)} = R^{(2)}(\mu).$$

Furthermore, Lemma 1 still holds for the discrete case. Hence,

$$J^{(3)}(\mu) - J^{(2)}(\mu) = \|f_*^{T_s}(lT_s)\|^2_{-M,M} - \|f_*^{N,T_s}(lT_s)\|^2_{-M,M} + 2\mu R^{(2)}(\mu) - 2\mu R^{(3)}(\mu). \tag{73}$$

It is easy to find from (73) and (70) that

$$\lim_{N\to\infty} J^{(3)}(\mu) = J^{(2)}(\mu).$$

Also, using the equation

$$R^{(2)}(\mu)T_s = \frac{1}{\mu^2} \sum_{l=-M}^{M} \sum_{n=-M}^{M} [g(lT_s) - f_*^{T_s}(lT_s)] \cdot [g(nT_s) - f_*^{T_s}(ns)] \frac{\sin \sigma\, T_s(l-n)}{\pi\, T_s(l-n)} T_s T_s$$

we obtain easily that

$$\lim_{T_s\to 0} R^{(2)}(\mu)T_s = \frac{1}{\mu^2} \int_{-T}^{T} \int_{-T}^{T} [g(t) - f_*(t)] \cdot [g(\tau) - f_*(\tau)] \frac{\sin \sigma(t-\tau)}{\pi(t-\tau)}\, dt\, d\tau = R^{(1)}(\mu).$$

Using Lemma 1, it follows that

$$J^{(2)}(\mu)T_s - J^{(1)}(\mu) = \int_{-T}^{T} |f_*(t)|^2\, dt - \sum_{l=-M}^{M} |f^{T_s}(lT_s)|^2\, T_s + 2\mu R^{(1)}(\mu) - 2\mu R^{(2)}(\mu)T_s + \sum_{l=-M}^{M} |g(lT_s)|^2\, T_s - \int_{-T}^{T} |g(t)|^2\, dt. \tag{74}$$

Thus, we conclude that

$$\lim_{T_s\to 0} J^{(2)}(\mu)T_s = J^{(1)}(\mu). \qquad \text{Q.E.D.}$$

Let $\mu_R^{(i)}, \mu_\epsilon^{(i)}$, $i = 1, 2, 3$ be the parameters μ_R and μ_ϵ, respectively, for the three cases mentioned above. Then, based on Theorem 5, we claim that

i) $$\lim_{N\to\infty} \mu^{(3)}_{R/\sqrt{T_s}} = \mu^{(2)}_{R/\sqrt{T_s}} \tag{75}$$

ii) $$\lim_{N\to\infty} \mu^{(3)}_{\epsilon/\sqrt{T_s}} = \mu^{(2)}_{\epsilon/\sqrt{T_s}} \tag{76}$$

iii) $$\lim_{T_s\to 0} T_s \cdot \mu^{(2)}_{R/\sqrt{T_s}} = \mu_R^{(1)} \tag{77}$$

iv) $$\lim_{T_s\to 0} T_s \cdot \mu^{(2)}_{\epsilon/\sqrt{T_s}} = \mu_\epsilon^{(1)} \tag{78}$$

where

$$R^{(3)}(\mu^{(3)}_{R/\sqrt{T_s}}) = R^2/T_s, \qquad J^{(3)}(\mu^{(3)}_{\epsilon/\sqrt{T_s}}) = \epsilon^2/T_s,$$
$$R^{(2)}(\mu^{(2)}_{R/\sqrt{T_s}}) = R^2/T_s, \qquad J^{(2)}(\mu^{(2)}_{\epsilon/\sqrt{T_s}}) = \epsilon^2/T_s,$$
$$R^{(1)}(\mu_R^{(1)}) = R^2, \qquad J^{(1)}(\mu_\epsilon^{(1)}) = \epsilon^2.$$

Therefore, $T_s \cdot \mu_R^{(3)}/\sqrt{T_s}$ and $T_s \cdot \mu_\epsilon^{(3)}/\sqrt{T_s}$ are the approximations of the parameters $\mu_R^{(1)}$ and $\mu_\epsilon^{(1)}$ when T_s is sufficiently small and N is sufficiently large.

In conclusion, we have proved that for $N\to\infty$ and $T_s\to 0$ the solution $f_*^{N,T_s}(mT_s)$ of the numerical implementation of the extrapolating algorithms will tend towards the analog solution $f_*(t)$. However, the important question of how large (small) should we choose N, (T_s) has not been addressed here.

VI. Numerical Example

We illustrate the method with a numerical example by solving Problem 1 for periodic band-limited sequences. The performed iteration has the following steps.

Step 1: Set

$$\mu_L = 0, \mu_U = \sum_{l=-M}^{M} |g[l]|^2/2R^2.$$

Step 2: Set

$$\mu = \frac{1}{2}(\mu_L + \mu_U),$$
$$f_0[l] = 0,$$
$$\alpha = 1/1 + \mu.$$

Step 3: Form

$$w_n[l] = (1 - \alpha\mu) f_{n-1}[l] + \alpha(g[l] - f_{n-1}[l])P_M[l].$$

Step 4: Take the DFS $W_n[m]$ of $w_n[l]$.

Step 5: Form

$$F_n[m] = \begin{cases} W_n[m] & |m| \leqslant K \\ 0 & |m| > K. \end{cases}$$

Step 6: Take the inverse DFS $f_n[l]$ of $F_n[m]$.

Step 7: Evaluate

$$R_n^2 = \sum_{l=0}^{N-1} |f_n[l]|^2 \qquad L_n = [1 - (1+\mu)^{-n}]^2 R^2.$$

Then, if

a) $R_n^2 > R^2$, set $\mu_U = \mu$ and go to step 2.

b) $R_n^2 < L_n$, set $\mu_L = \mu$ and go to step 2.

c) $R^2(1 - C) < R_n^2$, stop.

d) Otherwise, go to step 3.

In the example below, the computations were carried out with a DFS of size $N = 256$ and the various constants were set as follows:

$$M = 20 \qquad K = 15 \qquad C = 10^{-5}.$$

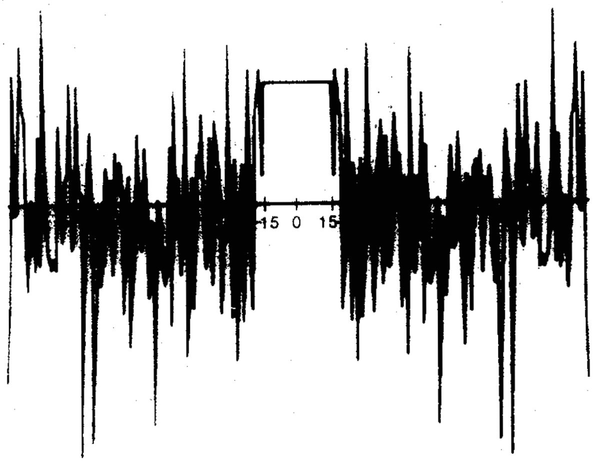
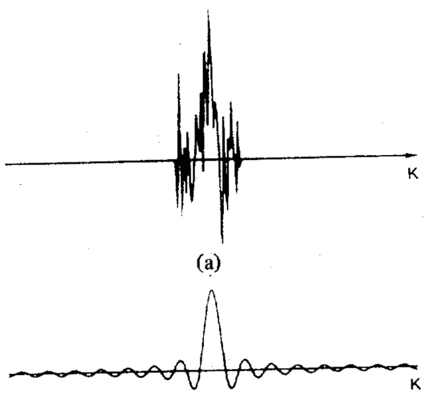


Fig. 2. $G[m]$: The DFS of the data.



(b)

Fig. 3. (a) The given data $g[l]$ for $|l| \leqslant 20$. (b) The inverse $f[l]$ of the K-band-limited part of $G[m]$.

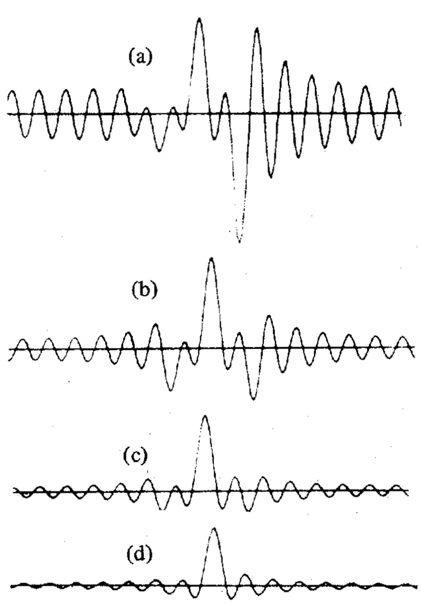


Fig. 4. The extrapolated $f_*[l]$ for (a) $\mu = 0$, $n = 500$. (b) $R^2 = 16.0$. (c) $R^2 = 256/31$. (d) $R^2 = 4.0$.

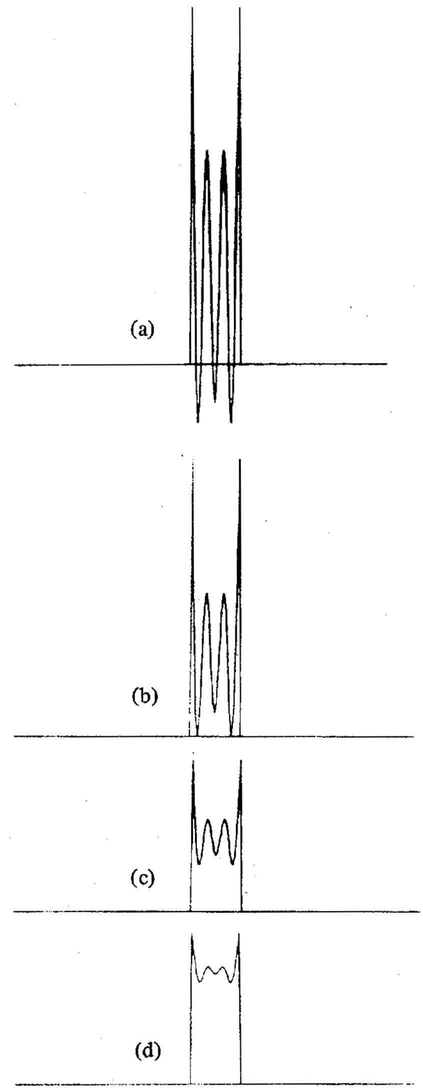


Fig. 5. The DFS $F_*[m]$ of $f_*[l]$ for (a) $\mu = 0$, $n = 500$. (b) $R^2 = 16.0$. (c) $R^2 = 256/31$. (d) $R^2 = 4.0$.

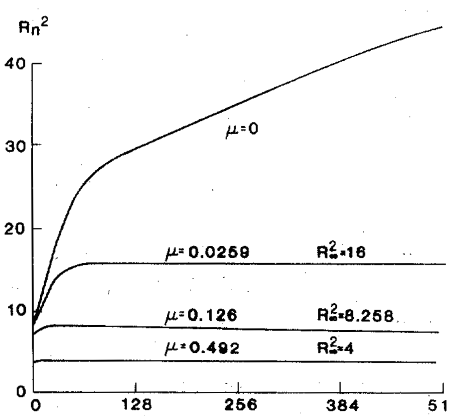


Fig. 6. R_n^2 for various values of μ versus n.

The data $g[l]$ were obtained from

$$G[m] = \begin{cases} \dfrac{1}{31} & |m| \leqslant K \\ \text{white noise} & \text{otherwise.} \end{cases}$$

$G[m]$ is shown in Fig. 2 and the given data $g[l]$ in Fig. 3(a). The unknown inverse $f[l]$ of the K-band-limited part of $G[m]$ is shown in Fig. 3(b). Its energy is 256/31. The extrapolation was performed for four different cases $\mu = 0, R^2 = 16.0, R^2 = 256/31$, $R^2 = 4.0$. The results are shown in Fig. 4, and their Fourier transforms in Fig. 5. It appears that $R^2 = 4.0$ gives us the best solution. This is because the solution $f_*[l]$ only can

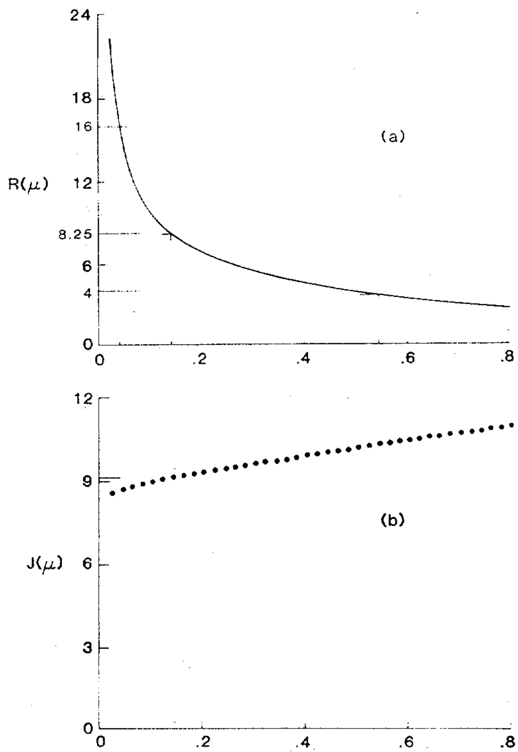


Fig. 7. The functions $R(\mu)$ and $J(\mu)$ for the presented example.

guarantee $\|f_*[l] - g[l]\|^2_{-M,M}$ is minimum over the family of K-band-limited N-periodic sequences whose energy is equal to or less than R^2. This does not imply that $\|f_*[l] - f[l]\|^2_{0,N-1}$ is minimum when $R^2 = \|f[l]\|^2_{0,N-1}$. The case $\mu = 0$ is the algorithm by Papoulis and Gerchberg. In Fig. 6, R_n^2 is shown for the four cases. It is clear that for $\mu = 0$ we have a diverging algorithm. Finally, in Fig. 7 the $R(\mu)$ and $J(\mu)$ of the given example are plotted as a function of μ. We obtain similar behavior from Problems 2 and 3.

VII. Conclusion

The applicability of the extrapolating algorithm proposed by Papoulis and Gerchberg was extended over noisy data by regularization. This was achieved by imposing energy constraints on the unknown band-limited signal or the noise. The discrete versions of the revised algorithm as well as the transition from the continuous algorithm to its digital implementation are presented. It is of interest to notice that by overcoming some difficulties on mathematics a similar algorithm can be derived if we consider $n(t)$ to be a stochastic process and seek for maximum likelihood solution. This result will be discussed in a future work.

Appendix A

The Prolate Spheroidal Functions

In this Appendix we simply state the definition of the prolate spheroidal function and the properties we need for the derivations in this work. For more details, see [1] or [11].

Definition: The prolate spheroidal functions are the eigenfunctions $\phi_k(t)$ of the equation

$$\int_{-T}^{T} \phi_k(\tau) \sin \sigma(t-\tau)/\pi(t-\tau)\, d\tau = \lambda_k \phi_k(t). \tag{A1}$$

Properties:

a) $\phi_k(t)$ are σ-band-limited, i.e.,

$$\phi_k(t) * \frac{\sin \sigma t}{\pi t} = \phi_k(t). \tag{A2}$$

b) $\phi_k(t)$ are orthonormal in $(-\infty, \infty)$ and orthogonal in $(-T, T)$, i.e.,

$$\int_{-\infty}^{\infty} \phi_k(t)\,\phi_l(t)\,dt = \begin{cases} 1 & k = l \\ 0 & k \neq l \end{cases} \tag{A3}$$

$$\int_{-T}^{T} \phi_k(t)\,\phi_l(t)\,dt = \begin{cases} \lambda_k & k = l \\ 0 & k \neq l \end{cases} \tag{A4}$$

c) The eigenvalues λ_k are such that

$$1 > \lambda_0 > \lambda_1 > \cdots > \lambda_k > \cdots > 0$$

$$\text{and} \quad \lim_{k \to \infty} \lambda_k = 0. \tag{A5}$$

d) Any σ-band-limited function $f(t)$ can be expressed as

$$f(t) = \sum_{k=0}^{\infty} a_k \phi_k(t)$$

$$a_k = \int_{-\infty}^{\infty} f(t)\,\phi_k(t)\,dt \tag{A6}$$

and

$$\|f(t)\|^2 = \sum_{k=0}^{\infty} a_k^2. \tag{A7}$$

e) Any function $g(t)$ with

$$\|g(t)\|_T^2 = \int_{-T}^{T} |g(t)|^2\,dt < \infty$$

can be expressed in the interval $(-T, T)$ as

$$g(t) = \sum_{k=0}^{\infty} b_k \phi_k(t) \qquad |t| < T$$

$$b_k = \frac{1}{\lambda_k} \int_{-T}^{T} g(t)\,\phi_k(t)\,dt \tag{A8}$$

and

$$\|g(t)\|_T^2 = \sum_{k=0}^{\infty} \lambda_k b_k^2. \tag{A9}$$

Appendix B

The Discrete Prolate Spheroidal Sequences (DPSS)

Definition: The discrete prolate spheroidal sequences are the eigenvectors $\phi_k[m]$ of the equation

$$\sum_{n=-M}^{M} \frac{\sin \sigma(m-n)}{\pi(m-n)} \phi_k[n] = \lambda_k \phi_k[m]$$

$$k = 0, \cdots, 2M. \tag{B1}$$

Properties:

a) $\phi_k[m]$ are σ-band-limited, i.e.,

$$\phi_k[m] * \frac{\sin \sigma m}{\pi m} = \phi_k[m]. \tag{B2}$$

b) $\phi_k[m]$ are doubly orthogonal, i.e.,

$$\sum_{m=-\infty}^{\infty} \phi_k[m]\,\phi_l[m] = \begin{cases} 1 & k=l \\ 0 & k \neq l \end{cases} \tag{B3}$$

$$\sum_{m=-M}^{M} \phi_k[m]\,\phi_l[m] = \begin{cases} \lambda_k & k=l \\ 0 & k \neq l. \end{cases} \tag{B4}$$

c) The eigenvalues λ_k are such that

$$1 > \lambda_0 > \lambda_1 > \cdots > \lambda_{2M} > 0.$$

d) There exist σ-band-limited sequences $\phi_k[m]$, $k = 2M+1, \cdots$, such that

$$\sum_{m=-\infty}^{\infty} \phi_k[m]\,\phi_l[m] = \begin{cases} 1 & k=l \\ 0 & k \neq l \end{cases}$$

$k, l = 0, 1, 2, \cdots$

and any σ-band-limited sequence $f[m]$ can be expressed as

$$f[m] = \sum_{k=0}^{\infty} a_k \phi_k[m]$$

$$a_k = \sum_{m=-\infty}^{\infty} f[m]\,\phi_k[m] \tag{B6}$$

and

$$\|f[m]\|^2 = \sum_{k=0}^{\infty} a_k^2. \tag{B7}$$

e) Any vector $(g[-M], \cdots, g[M])$ can be expressed as

$$g[m] = \sum_{k=0}^{2M} b_k \phi_k[m] \qquad |m| \leqslant M$$

$$b_k = \frac{1}{\lambda_k} \sum_{m=-M}^{M} g[m]\,\phi_k[m] \qquad k = 0, \cdots, 2M \tag{B8}$$

and

$$\|g[m]\|^2_{-M,M} = \sum_{k=0}^{2M} \lambda_k b_k^2. \tag{B9}$$

It is clear that if $g[m] = f[m]$ for $|m| \leqslant M$, $b_k = a_k$ for $k = 0, 1, \cdots, 2M$.

Appendix C
The Periodic Discrete Prolate Spheroidal Sequences (P-DPSS)

Definition: The periodic discrete prolate spheroidal sequences are the eigenvectors $\phi_i[m]$ of the equation, i.e.,

$$\sum_{n=-M}^{M} \frac{\sin \frac{\pi}{N}(2K+1)(m-n)}{N \sin \frac{\pi}{N}(m-n)}\,\phi_i[n] = \lambda_i \phi_i[m]$$

$$i = 0, \cdots, 2M \tag{C1}$$

where

$$2K+1 < N, \quad 2M+1 < N.$$

Properties:

a) The eigenvalues λ_i are such that

$$1 \geqslant \lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_{2M} > 0 \quad \text{if} \quad M \leqslant K \tag{C2}$$

or

$$1 \geqslant \lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_{2K} > \lambda_{2K+1} = \cdots = \lambda_{2M} = 0$$
$$\text{if } M > K. \tag{C3}$$

b) $\phi_i[m]$ are doubly orthogonal, i.e.,

$$\sum_{m=l_0}^{l_0+N-1} \phi_i[m]\,\phi_j[m] = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \tag{C4}$$

$$\sum_{m=-M}^{M} \phi_i[m]\,\phi_j[m] = \begin{cases} \lambda_i & i=j \\ 0 & i \neq j \end{cases} \tag{C5}$$

where

$$i, j = 0, \cdots, \min(2K, 2M). \tag{C6}$$

c) $\phi_i[m], i = 0, \cdots, \min(2K, 2M)$, are K-band-limited, i.e.,

$$\sum_{n=l_0}^{l_0+N-1} \frac{\sin \frac{\pi}{N}(2K+1)(m-n)}{N \sin \frac{\pi}{N}(m-n)}\,\phi_i[n] = \phi_i[m]. \tag{C7}$$

d) When $M \leqslant K$, there exist K-band-limited N-periodic sequences $\phi_i[m]$, $i = 2M+1, \cdots, 2K$, such that

$$\sum_{m=l_0}^{l_0+N-1} \phi_i[m]\,\phi_j[m] = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

and any K-band-limited N-periodic sequence $f[m]$ can be expressed as

$$f[m] = \sum_{i=0}^{2K} a_i\,\phi_i[m]$$

$$a_i = \sum_{m=l_0}^{l_0+N-1} f[m]\,\phi_i[m] \tag{C8}$$

and

$$\|f[m]\|^2_{l_0, l_0+N-1} = \sum_{i=0}^{2K} a_i^2. \tag{C9}$$

e) When $M > K$, any K-band-limited N-periodic sequence $f[m]$ can be expressed as

$$f[m] = \sum_{i=0}^{2K} a_i \phi_i[m]$$

$$a_i = \sum_{m=l_0}^{l_0+N-1} f[m]\, \phi_i[m] \tag{C10}$$

and

$$\|f[m]\|^2_{l_0, l_0+N-1} = \sum_{i=0}^{2K} a_i^2 . \tag{C11}$$

f) When $M \leqslant K$, any vector $(g[-M], \cdots, g[M])$ can be expressed as

$$g[m] = \sum_{i=0}^{2M} b_i \phi_i[m] \qquad |m| \leqslant M$$

$$b_i = \frac{1}{\lambda_i} \sum_{m=-M}^{M} g[m]\, \phi_i[m] \tag{C12}$$

and

$$\|g[m]\|^2_{-M,M} = \sum_{i=0}^{2M} \lambda_i b_i^2 . \tag{C13}$$

g) When $M > K$, any vector $(g[-M], \cdots, g[M])$ can be expressed as

$$g[m] = \sum_{i=0}^{2K} b_i \phi_i[m] + \tilde{g}[m] \qquad |m| \leqslant M$$

$$\sum_{m=-M}^{M} \tilde{g}[m]\, \phi_i[m] = 0 \qquad i = 0, \cdots, 2K \tag{C14}$$

$$b_i = \frac{1}{\lambda_i} \sum_{m=-M}^{M} g[m]\, \phi_i[m] \tag{C15}$$

and

$$\|g[m]\|^2_{-M,M} = \sum_{i=0}^{2K} \lambda_i b_i^2 + \|\tilde{g}[m]\|_{-M,M} . \tag{C16}$$

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References

[1] D. Slepian, H. O. Pollak, and H. J. Landau, "Prolate spheroidal wave functions I, II," *Bell Syst. Tech. J.*, vol. 40, pp. 43–84, Jan. 1961.
[2] A. Papoulis, "A new method of image restoration," JSTAC, Contract F44620-74-C-0056, Prog. Rep. 39, Rep. R-452, pp. 39–74, Paper VT-3, 1973–1974.
[3] R. W. Gerchberg, "Super-resolution through error energy reduction," *Opt. Acta*, vol. 21, no. 9, pp. 709–720, 1974.
[4] A. Papoulis, "A new algorithm in spectral analysis and band-limited extrapolation," *IEEE Trans. Circuits Syst.*, vol. CAS-22, pp. 735–742, Sept. 1975.
[5] P. De Santis and F. Gori, "On an iterative method for super resolution," *Opt. Acta*, vol. 22, no. 8, pp. 691–695, 1975.
[6] D. C. Youla, "Generalized image restoration by the method of alternating orthogonal projections," *IEEE Trans. Circuits Syst.*, vol. CAS-25, pp. 694–701, Sept. 1978.
[7] G. A. Viano, "On the extrapolation of optical image data," *J. Math. Phys.*, vol. 17, pp. 1160–1165, July 1976.
[8] K. Miller, "Least-square methods for ill-posed problems with a prescribed bound," *SIAM Rev.*, vol. 1, pp. 52–74, Feb. 1970.
[9] M. Bertero, G. A. Viano, and C. DeMol, "Resolution beyond the Rayleigh limit for regularizing object restoration," *Opt. Acta*, vol. 27, pp. 307–320, 1980.
[10] C. K. Rushford and R. L. Frost, "Comparison of some algorithms for reconstructing space-limited images," *J. Opt. Soc. Amer.*, vol. 70, p. 1539, 1980.
[11] A. Papoulis, *Signal Analysis.* New York: McGraw-Hill, 1977.
[12] A. Tikhonor and V. Arsenin, *Solutions of Ill-Posed Problems.* New York: Wiley, 1977.
[13] J. A. Cadzow, "An extrapolation procedure for band-limited signals," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-27, pp. 4–12, Feb. 1979.
[14] D. Slepian, "Prolate spheroidal wave functions, Fourier analysis and uncertainty—V: The discrete case," *Bell Syst. Tech. J.*, vol. 57, pp. 1371–1430, May–June 1978.
[15] A. Papoulis and M. Betran, "Digital filtering and prolate functions," *IEEE Trans. Circuits Syst.*, vol. CAS-11, pp. 674–681, Nov. 1972.
[16] C. Chamzas, "On the extrapolation of band-limited functions," Ph.D. dissertation, Dep. Elec. Eng., Polytech. Inst. New York, Farmingdale, June 1980.
[17] A. K. Jain and S. Ranganath, "Extrapolation algorithms for discrete signals with application in spectral estimation," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-29, pp. 830–845, Aug. 1981.
[18] M. S. Sabri and W. Steenaart, "An approach to band-limited signal extrapolation: The extrapolation matrix," *IEEE Trans. Circuits Syst.*, vol. CAS-25, pp. 74–78, Feb. 1978.
[19] S. G. Miklein, *Integral Equations.* New York: Macmillan, 1964.

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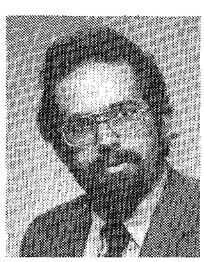

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