TABLE I
RESULTS OF ANALYTICAL AND EXPERIMENTAL α_0 DETERMINATION
($A = \exp[-\tau_v]$, $B = \exp[-\tau_h]$)

$\sigma^2 = 10$ (A, B)	Analytical α_0	Experimental α_0
(0.95, 0.70)	$4(10)^{-4}$	$4(10)^{-4}$
(0.70, 0.95)	$2(10)^{-4}$	$2(10)^{-4}$
(0.95, 0.90)	$0.9(10)^{-4}$	$1(10)^{-4}$
(0.90, 0.95)	$0.3(10)^{-4}$	$2(10)^{-4}$

TABLE II
RESULTS OF ANALYTICAL AND EXPERIMENTAL ϵ_0 DETERMINATION

$\sigma^2 = 10$ (A, B)	Analytical ϵ_0, dB	Experimental ϵ_0, dB
(0.95, 0.70)	−3.84	−3.77
(0.70, 0.95)	−2.08	−1.80
(0.95, 0.90)	−0.93	−1.01
(0.90, 0.95)	−0.51	−0.63

analytical techniques developed. In practice, one may not have an absolute knowledge of the data statistical parameters. However, a good knowledge of statistical bounds may be known *a priori*. In this case, a range of α_0 values may be obtained (along with a range of expected ASE reductions, the ϵ_0) from which a representative α_0 might be selected. It should be noted from the results in Table I that even though statistical parameters may vary widely, the corresponding α_0 may not vary in an extreme manner. This is illustrated by the two cases in Table I, $(A, B) = (0.95, 0.70)$ and $(A, B) = (0.70, 0.95)$ which have drastically differing correlation properties, but have α_0 of $4(10)^{-4}$ and $2(10)^{-4}$, respectively.

IV. SUMMARY AND CONCLUSION

This paper has presented a method for analytically deriving α_0, the optimum LMS gain parameter for use with a visual fidelity criterion for image source coding. This paper has shown that if one has knowledge of image correlation parameters then α_0 may be calculated directly. If one only has knowledge of an expected range of input statistical parameters, then one may correspondingly calculate a range of α_0 which will provide near minimum ASE.

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An Improved Version of Papoulis–Gerchberg Algorithm on Band-Limited Extrapolation

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***Abstract*—An iterative algorithm for extrapolating analog band-limited signals has been proposed by Papoulis and Gerchberg.[1] By inserting a multiplication by a constant in the above algorithm, chosen to minimize the energy of the error in the extrapolation interval, a considerable speed up of its convergence has been achieved.**

I. INTRODUCTION

A central problem in Fourier analysis and spectral estimation is the determination of the transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

of a signal $f(t)$ in terms of a finite segment

$$f_T(t) = f(t)\, P_T(t) \tag{1}$$

where

$$P_T(t) = \begin{cases} 1 & |t| \leqslant T \\ 0 & |t| > T. \end{cases}$$

If, in addition of $f_T(t)$, it is known that $f(t)$ is σ-band-limited, i.e.,

$$F(\omega) = F(\omega)\, P_\sigma(\omega) \tag{2}$$

$$E^2 = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} |F(\omega)|^2\, d\omega < \infty$$

and

$$P_\sigma(\omega) = \begin{cases} 1 & |\omega| < \sigma \\ 0 & \text{otherwise,} \end{cases}$$

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[1]The band-limited extrapolation algorithm was first presented by A. Papoulis in 1973 in [2]; it was published independently by R. W. Gerchberg in 1974 [3]; the proof of the convergence was given by A. Papoulis in 1975 [4].

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then it is well known that $f(t)$ is analytic in the entire t-axis [1], and therefore, in principle, $f(t)$ can be recovered from its segment $f_T(t)$. However, extrapolating $f_T(t)$ by using Taylor series expansion is impractical because derivative is a noise sensitive operation. In [2]-[5], an algorithm was developed, where the segment $f_T(t)$ is extrapolated iteratively over the entire t-axis. Although the suggested algorithm is less sensitive to noise, noise is still a major problem and the iteration is diverging when the given segment is contaminated with non-band-limited noise. Papoulis [4] has suggested for solution its early termination, but the determination of a termination criterion is not available yet. A different approach is used in [9] where the problem of extrapolating noisy data is solved by assuming that energy constraints are known either for the band-limited signal or for the noise.

Another problem associated with iterative algorithms is the speed of convergence. In the present work we consider only the second problem and we propose a simple modification of the original algorithm, which speeds up considerably its convergence.

In the original algorithm, at the nth iteration step, $f(t)$ is approximated by

$$f_n(t) = w_{n-1}(t) *^2 \frac{\sin \sigma t}{\pi t}$$

with

$$w_{n-1}(t) = \begin{cases} f_T(t) & \text{for } |t| \leqslant T \\ f_{n-1}(t) & \text{for } |t| > T \end{cases} \tag{3}$$

$$f_0(t) = 0.$$

In the proposed modified version, the nth iteration step becomes

$$f_n(t) = w_{n-1}(t) * \frac{\sin \sigma t}{\pi t}$$

where

$$w_{n-1}(t) = \begin{cases} f_T(t) & \text{for } |t| \leqslant T \\ A_{n-1} f_{n-1}(t) & \text{for } |t| > T. \end{cases} \tag{4}$$

with

$$f_0(t) = 0.$$

The constant A_n is chosen to minimize the energy of the error, I_n, in the nth iteration step. The definition of I_n depends on what we consider as the end of the nth step. If $f_n(t)$ is the nth approximation of $f(t)$, then

$$I_n = \int_{-\infty}^{\infty} |f(t) - A_n f_n(t)|^2 \, dt. \tag{5}$$

If $w_n(t)$ is the nth approximation of $f(t)$, then

$$I_n = \int_{-\infty}^{\infty} |f(t) - A_n f_n(t)|^2 \, (1 - P_T(t)) \, dt. \tag{6}$$

The use of the above technique to the discrete version of the algorithm, i.e., extrapolating σ-band-limited sequences or periodic band-limited sequences, is easy and will not be described here. The formulas for the evaluation of A_n are obtained by simply replacing integrations by summations and the proof of convergence is simpler. Similar results for the discrete algorithms, have been already reported in a detailed treatment by Jain and Ranganath, [6], where also the question of uniqueness was considered.

We want to emphasize, that by discrete version, we do not mean the numerical implementation of the continuous algorithm. The problems related with the numerical implementation will not be considered in this work. The problem has been examined in [9].

[2] (*) denotes convolution.

II. Evaluation of A_n

We shall use relation (5) to derive A_n. Similar results are obtained if (6) is used and they are given at the end of this section.

Theorem 1:

$$A_n = \frac{X_n}{W_n} \tag{7}$$

with

$$W_n = \int_{-\infty}^{\infty} |f_n(t)|^2 \, dt \tag{8}$$

$$Y_n = \int_{-T}^{T} f(t) f_n^*(t) \, dt$$

$$X_n = \int_{-\infty}^{\infty} f(t) f_n^*(t) \, dt$$

$$= X_1 + A_{n-1}(X_{n-1} - Y_{n-1}) \tag{9}$$

where

$$X_1 = \int_{-T}^{T} |f(t)|^2 \, dt \quad A_0 = 1, \quad X_0 = Y_0 = 0.$$

Proof: From (6), using the orthogonality principle, we obtain

$$\int_{-\infty}^{\infty} (f(t) - A_n f_n(t)) f_n^*(t) \, dt = 0 \tag{10}$$

and (7) follows.

Since $f(t)$ is given for $|t| < T$ and $f_n(t)$ is known in each iteration step, W_n and Y_n can be evaluated directly. For the evaluation of X_n, we have to prove the recursive relation (9). In the following steps we will assume $f(t)$ to be real, however the result is valid even if $f(t)$ is complex [7].

From (4), we have

$$f_n(t) = [f(t) P_T(t) + A_{n-1}(1 - P_T(t)) f_{n-1}(t)] * \frac{\sin \sigma t}{\pi t}. \tag{11}$$

a) Evaluation of X_1:

$$X_1 = \int_{-\infty}^{\infty} f(t) f_1(t) \, dt = \int_{-\infty}^{\infty} f(t) \int_{-\infty}^{\infty} f(\tau) P_T(\tau) \cdot \frac{\sin \sigma(t-\tau)}{\pi(t-\tau)} \, d\tau \, dt$$

$$= \int_{-\infty}^{\infty} f(\tau) P_T(\tau) \int_{-\infty}^{\infty} f(t) \frac{\sin \sigma(t-\tau)}{\pi(t-\tau)} \, dt \, d\tau.$$

But $f(t)$ is σ-band-limited and

$$\int_{-\infty}^{\infty} f(t) \frac{\sin \sigma(t-\tau)}{\pi(t-\tau)} dt = f(\tau). \quad (12)$$

Hence,

$$X_1 = \int_{-\infty}^{\infty} |f(\tau)|^2 P_T(\tau)\, d\tau = \int_{-T}^{T} |f(\tau)|^2\, d\tau.$$

Therefore, X_1 is known.

b) Evaluation of X_n:

$$\begin{aligned} X_n &= \int_{-\infty}^{\infty} f(t) f_n(t)\, dt = \int_{-\infty}^{\infty} f(t) \int_{-\infty}^{\infty} [f(\tau) P_T(\tau) \\ &\quad + A_{n-1}(1 - P_T(\tau)) f_{n-1}(\tau)] \frac{\sin \sigma(t-\tau)}{\pi(t-\tau)} d\tau\, dt \\ &= \int_{-\infty}^{\infty} f(t) \int_{-\infty}^{\infty} f_T(\tau) \frac{\sin \sigma(t-\tau)}{\pi(t-\tau)} d\tau\, dt + A_{n-1} \int_{-\infty}^{\infty} \\ &\quad \cdot f(t) \int_{-\infty}^{\infty} f_{n-1}(\tau)(1 - P_T(\tau)) \frac{\sin \sigma(t-\tau)}{\pi(t-\tau)} d\tau\, dt \\ &= X_1 + A_{n-1}\left[\int_{-\infty}^{\infty} f(t) f_{n-1}(\tau)\, d\tau \right. \\ &\quad \left. - \int_{-\infty}^{\infty} f_T(t) f_{n-1}(\tau) d\tau\right] \\ &= X_1 + A_{n-1}(X_{n-1} - Y_{n-1}). \end{aligned}$$

Q.E.D.

Thus, X_n and A_n can be evaluated recursively in each step.

Note: If relation (6) is used, then

$$A_n = \frac{X_n - Y_n}{Z_n} \quad (13)$$

where X_n and Y_n are given in (8), and

$$Z_n = \int_{|t|>T} |f_n(t)|^2\, dt. \quad (14)$$

To illustrate the significance of the constant, we present a simple example.

Example: Let us assume that the unknown function is

$$f(t) = \phi_k(t)$$

where $\phi_k(t)$ is one of the prolate spheroidal functions (see [1, p. 205]).

a) If the original iteration ($A_n = 1$), is applied, then, as it has been shown in [4]

$$f_n(t) = [1 - (1 - \lambda_k)^n]\, \phi_k(t) \qquad 0 < \lambda_k < 1$$

b) If A_n is used, then from (4) and (7) we obtain that

$$f_1(t) = \lambda_k \phi_k(t), \quad X_1 = \lambda_k \quad W_1 = \lambda_k^2 \quad A_1 = 1/\lambda_k$$

and

$$w_1(t) = \begin{cases} \phi_k(t) & |t| < T \\ \dfrac{1}{\lambda_k} \lambda_k \phi_k(t) & |t| > T \end{cases} = \phi_k(t)$$

Hence, we need *only one* iteration to recover $\phi_k(t)$ from its segment $\phi_k(t) P_T(t)$, instead of the infinite many needed in part *a*).

III. Convergence

In this section, we consider the question of convergence. We can prove that

$$\lim_{n\to\infty} \|f - f_n\| = 0 \quad (15)$$

where

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(t)|^2\, dt.$$

We can also prove the following more general theorem on convergence.

Theorem 2: Let $f(t)$ be a σ-band-limited function (2), and let

$$f_n(t) = [f(t) P_T(t) + A_{n-1}(1 - P_T(t)) f_{n-1}(t)] * \frac{\sin \sigma t}{\pi t}$$

where A_n is a positive sequence of numbers. Then

$$\lim_{n\to\infty} f_n(t) = f(t)$$

if and only if

$$\lim_{n\to\infty} A_n = 1 \quad \text{and} \quad \lim_{n\to\infty} \|f_n\| = \|f\|.$$

The proofs are not presented, due to space limitations. However, they are available upon request.

Convergence can also be proved if A_n is defined by minimizing I_n in (6) instead of (5). Moreover, it is easy to generalize [7] the above method for signals which are bandpass limited and either one or more segments are known anywhere over the t axis.

To avoid the difficulties associated with the infinite limits of integration in the Fourier integral, we can use a form for the iteration similar to the one suggested by Cadzow [8], i.e.,

$$\begin{aligned} f_n(t) &= A_{n-1} f_{n-1}(t) + \int_{-T}^{T} [f(\tau) - A_{n-1} f_{n-1}(\tau)] \\ &\quad \cdot \frac{\sin \sigma(t-\tau)}{\pi(t-\tau)} d\tau \end{aligned}$$

$$f_0(t) = 0$$

where A_n is evaluated as in Theorem 1. Notice that A_n needs, also, only evaluation of integrals with finite limits, because W_n can be derived recursively by

$$\begin{aligned} W_n &= A_{n-1}^2 W_{n-1} + 2A_{n-1} \int_{-T}^{T} f_{n-1}(\tau) [f(\tau) \\ &\quad - A_{n-1} f_{n-1}(\tau)]\, d\tau + \int_{-T}^{T}\int_{-T}^{T} [f(\tau) - A_{n-1} f_{n-1}(\tau)] \\ &\quad \cdot [f(\rho) - A_{n-1} f_{n-1}(\rho)] \frac{\sin \sigma(\tau-\rho)}{\pi(\tau-\rho)} d\tau\, d\rho \end{aligned}$$

and $W_0 = 0$.

IV. Conclusion

A steepest descent technique has been used to speed considerably the convergence of the Papoulis–Gerchberg algorithm for the extrapolation of band-limited functions. A recursive relation has been obtained for the evaluation of the optimum constant, and the convergence of the modified algorithm has been proven for the continuous noise free case. The problem of extrapolating noisy data is considered by the authors in another work [9].

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Comments and Corrections on "On the Eigenvectors of Symmetric Toeplitz Matrices"

LOKESH DATTA AND SALVATORE D. MORGERA

Abstract—The necessary and sufficient condition for a matrix to be doubly symmetric, as stated in the above paper,[1] is questioned. It is proved that the condition under consideration is necessary but *not* sufficient. Some counterexamples are provided to substantiate the claim. In fact, what appears to be a new class of matrices possessing some interesting properties is discussed.

In the above paper,[1] it is shown that if Q is an $N \times N$ doubly-symmetric matrix and J is the $N \times N$ reflection or contraidentity matrix having ones along the cross diagonal and zeroes elsewhere, then (9a) and (9b) state

$$JQ = QJ \tag{1}$$

and

$$JQJ = Q \tag{2}$$

respectively. Moreover, it is claimed that (2) is the necessary and sufficient condition for Q to be doubly symmetric. We disagree with the sufficiency condition of (2), the proof for which is as follows.

Let the elements of Q be $\{q_{ij} | 1 \leq i, j \leq N\}$. Then, using (2), we obtain

$$q_{ij} = q_{N+1-i, N+1-j} \tag{3}$$

which neither implies

$$q_{ij} = q_{ji} \qquad \text{(symmetry)} \tag{4}$$

nor

$$q_{ij} = q_{N+1-j, N+1-i} \qquad \text{(persymmetry).} \tag{5}$$

The equalities (4) and (5) must both hold for the matrix Q to be doubly symmetric, a property which cannot be inferred from (3) alone. The fact is that (3) does not imply any relationship between i and j. Therefore, a matrix satisfying (2) may or may not be doubly symmetric.

We now present three examples of matrices which satisfy (2), but are not doubly symmetric.

Example 1: $N = 3$

$$Q = \begin{bmatrix} 1 & -3 & 2 \\ 5 & -9 & 5 \\ 2 & -3 & 1 \end{bmatrix}$$

Example 2: $N = 4$

$$Q = \begin{bmatrix} 8 & 6 & 4 & 3 \\ 2 & 7 & 5 & 1 \\ 1 & 5 & 7 & 2 \\ 3 & 4 & 6 & 8 \end{bmatrix}$$

Example 3: $N = 5$

$$Q = \begin{bmatrix} 1 & 2 & 5 & 4 & 3 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 12 & 11 \\ 10 & 9 & 8 & 7 & 6 \\ 3 & 4 & 5 & 2 & 1 \end{bmatrix}.$$

Since the sufficiency of (2) has been disproved, a comment on the proof of Property 1 as stated in the paper is in order. Property 1 states that the inverse of a nonsingular doubly-symmetric matrix is doubly symmetric. The proof should be concluded by stating that since $JQ^{-1}J = Q^{-1}$ and Q^{-1} is symmetric, Q^{-1} is doubly symmetric.

Let us now discuss briefly the structure of matrices that satisfy (2). It is easy to prove that a matrix Q satisfying $JQJ = Q$ can be partitioned as follows for even and odd order:

i) $N = 2M$ (even order):

$$Q = \left[\begin{array}{c|c} A & B \\ \hline JBJ & JAJ \end{array}\right]$$

ii) $N = 2M + 1$ (odd order):

$$Q = \left[\begin{array}{c|c|c} A & s & B \\ \hline t^T & \rho & t^TJ \\ \hline JBJ & Js & JAJ \end{array}\right]$$

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