DETECTION

General problem of detection:

A signal x(t) is received : x(t) = $\begin{cases} n(t) \\ s(t)+n(t) \end{cases}$, n(t) is the noise and s(t) is the signal and we know it precisely.

The observer must choose between two possible situations

- H_0 : There is no signal and x(t) is just noise
- H_1 : The signal is present and x(t)=s(t)+n(t)

Assumptions:

(1) The signal x(t) is known and we want to know if it is present or not.

(2) n(t) is a stochastic process with known statistics.

(3) The observer must base its decision on

   a. Samples of x(t) (digital x(nT))
   b. The entire x(t) (analog)

We shall address the digital problem:

We assume that (a) s(t) and n(t) are independent

(b) The samples of n(t) are uncorrelated

Objectives:

Design the optimum system:

- The system can be linear or nonlinear
- System means → Hardware device
  → Software program

Criteria:

The system must be optimum with respect to a criterion which depends on the particular problem.

Sometimes we are satisfied with a sub optimum system because it is much simpler than the optimum.

Parametric detection: if we know the p.d.f of n(t) (complete description)

Non parametric detection: if we don't have a complete description of n(t).

(ex. : we only know $R_n[n] = \sigma_n^2 \delta[n]$)

Robust detector: is called a detector, which performs within acceptable levels even if the assumption regarding noise is violated or otherwise performs equally, well in a different environment than the assumed one.

Example:

Assume that noise is Gaussian and then investigate its performance under noise with another distribution.

Errors:

Due to the presence of noise we can perform errors in our decisions

(1). Error of the first kind: Pe_0. It is the probability to decide that the signal is present when it is not or choose H_1 when H_0 is true.

(2). Error of the second kind: Pe_1. When the signal is present you decide that it is not or choose H_0 when H_1 is true.

Problem 1

Detection of a pulse (PCM or radar e.t.c)

Case A

Assumption:

a. $s(t) = \begin{cases} A \\ 0 \end{cases}$
b. Probability of s(t) = A = P_1
c. Probability of s(t) = 0 = 1-P_1
d. Probability density function of n(t) is known
e. We must base our decision on one sample

Criterion: Minimize the probability of error:

$$Pe = P_1 Pe_1 + (1-P_1) Pe_0$$

Solution: Decision will be based on $x[n]=x_1$. If x_1 is in one Range R_1 then we decide that H_1 is true and our problem is to find R_1.

$$P_{e0} = \{x_1 \text{ is in } R_1 \text{ but s(t) is not present}\} = \int_{R_1} f_{x_1}(x_1 / s(t) = 0) dx$$

$$P_{e1} = \int_{1-R_1} f_{x_1}(x_1 / s(t) = A)dx = 1 - \int_{R_1} f_{x_1}(x_1 / s(t) = A)dx$$

$\Rightarrow$ Total Error

$$P_e = (1-p_1)\int_{R_1} f_{x_1}(x_1 / s(t) = 0)dx + p_1\left(1 - \int_{R_1} f_{x_1}(x_1 / s(t) = A)dx\right) =$$

$$= p_1 + \int_{R_1}[(1-p_1)f_{x_1}(x/0) - p_1 f_{x_1}(x/A)]dx$$

Choose R_1 such that

$$[(1-P1)f_{x_1}(x/0)] - P_1 f_{x_1}(x/A)] < 0$$

$$l(x_1) = \frac{f_{x_1}(x/A)}{f_{x_1}(x/0)} > \frac{1-P_1}{P_1}$$

$l(x_1)$ = Likelihood radio

Sometimes it is more convenient to take logarithms

$$\ln l(x_1) > \ln\frac{1-P_1}{P_1}$$

Comment: the worst case will be Pe = 1/2

Sub case A.1

Noise is gaussian distributed with zero mean and variance σ^2

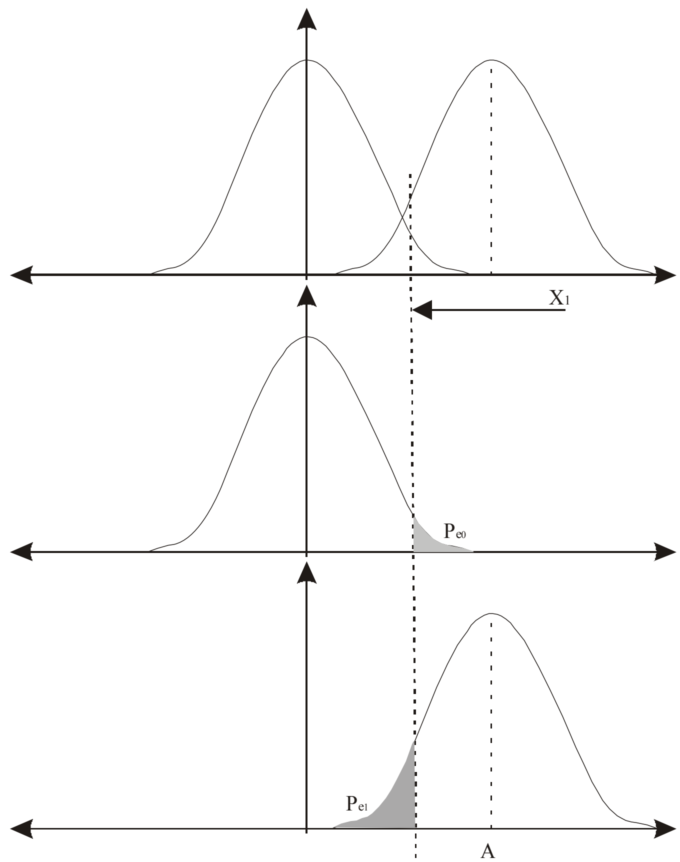
$$\frac{\frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x_1-A)^2}{2\sigma_x^2}}}{\frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-x_1^2}{2\sigma_x^2}}} \geq \frac{1-P_1}{P_1} \Rightarrow \left(\frac{-(x_1-A)^2 + x_1^2}{2\sigma_x^2}\right) \geq \ln\frac{1-P_1}{P_1} \Rightarrow \frac{x_1 A}{\sigma_x^2} - \frac{A^2}{2\sigma_x^2} \geq \ln\frac{1-P_1}{P_1}$$

$$\Rightarrow X_1 > \frac{A}{2} + \frac{\sigma_x^2}{A}\ln\frac{1-P_1}{P_1} = D$$

Ορισμός: $erf(x) = \frac{1}{\sqrt{\pi}}\int_{-x}^{x} e^{-t^2}dt = \frac{2}{\sqrt{\pi}}\int_{0}^{x} e^{-t^2}dt$

In this case probability of error is:

$$Pe = \frac{1}{2} - \frac{1}{2}\left(P_1 erf\frac{A-D}{\sqrt{2}\sigma_x} + (1-P_1)erf\frac{D}{\sqrt{2}\sigma_x} \right)$$



If P_1=1/2 then we have the expected decision x_1>A/2 signal is present.

<u>Numerical example:</u>

Computer transmits 0,1,… Each pulse occupies T sec and we sample every T sec.

$P_1=P_2=1/2$ choose A such that $Pe=10^{-5}$.

$$D = \frac{A}{2} \Rightarrow Pe = \frac{1}{2} - \frac{1}{4}\cdot\left[2erf\frac{A}{2\cdot 2}\right] = \frac{1}{2} - \frac{1}{2}\cdot\left[erf\frac{A}{4}\right]$$

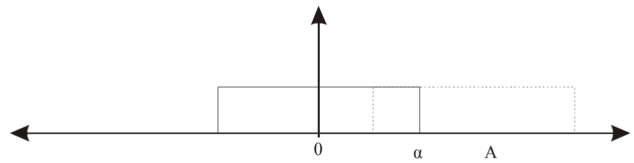
$$\Rightarrow erf\frac{A}{4} = 2\cdot\left[\frac{1}{2} - Pe\right] = 0.99998 \Rightarrow \frac{A}{4} = 3 \Rightarrow A = 12$$

Let T=1μs $\Rightarrow$ on the average we are going to perform 10 errors every second!

Sub case A.2

Assume a different distribution

Uniform distribution



Pe=0 if $A-a>a$, R1=$x \geq A-a$

$A-a<a$ or $A<2a$

$$l(x)=\begin{cases}0 & x<A-a\\ 1 & A-a<x<a\\ \infty & a<x\end{cases}$$